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Renormalization of the long-wavelength solution of Einstein's equation

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Using the renormalization group method, we improve the first order solution of the long-wavelength expansion of the Einstein equation. By assuming that the renormalization group transformation has the property of a Lie group, we can regularize the secular divergence caused by the spatial gradient terms and absorb it into the background seed metric. The solution of the renormalization group equation shows that the renormalized metric describes the behavior of gravitational collapse in the expanding universe qualitatively well. [S0556-2821(99)10018-3]

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I. INTRODUCTION

The spatial gradient expansion [1–4] of the Einstein equation is a nonlinear approximation method which describes the long-wavelength inhomogeneity in the universe. This approximation scheme expands Einstein's equation by the order of the spatial gradient. As a background solution, we solve Einstein's equation by neglecting all spatial gradient terms. The resultant solution has the same form as that for the spatially flat Friedman-Robertson-Walker (FRW) universe, but the three-metric can have a spatial dependence. It is possible to include the effect of spatial gradient terms by calculating the next order. This method can describe a long-wavelength nonlinear perturbation without imposing any symmetry for a space, and is suitable for analyzing the global structure of an inhomogeneous universe. Furthermore, it is easy to include a perfect fluid with pressure and scalar fields.

But this scheme is valid only for a perturbation whose wavelength is larger than the Hubble horizon scale. For a matter field which satisfies the energy conditions, the perturbation terms induced by the spatial gradient terms grows in time and finally dominates the background solution. This occurs when the wavelength of the perturbation equals the Hubble horizon scale. After this time, the wavelength of the perturbation becomes shorter than the horizon and the result of the gradient expansion becomes unreliable.

A similar situation occurs in the field of nonlinear dynamical systems. To obtain temporal evolution of the solution of a nonlinear differential equation, we usually apply a perturbative expansion. But naive perturbation often yields secular terms due to resonance phenomena. The secular terms prevent us from getting approximate but global solutions. There are many techniques to circumvent the problem, for example, the averaging method, multitime scale method, WKB method, and so on [5,6]. Although these methods, yield globally valid solutions, they provide no systematic procedure for general dynamical systems because we must select a suitable assumption on the structure of the perturbation series.

The renormalization group method [7–11] as a tool for global asymptotic analysis of the solution to differential equations unifies the techniques listed above, and can treat many systems irrespective of their features. Starting from a naive perturbative expansion, the secular divergence is absorbed into the constants of integration contained in the zeroth order solution by the renormalization procedure. The renormalized constants obey the renormalization group equation. This method can be viewed as a tool of system reduction. The renormalization group equation corresponds to the amplitude equation which describes slow motion dynamics in the original system. We can describe complicated dynamics contained in the original equation by extracting a simpler representation using the renormalization group method.

In this paper, we apply the renormalization group method to the gradient expansion of Einstein's equation. Our purpose is to obtain the renormalized long-wavelength solution of Einstein's equation which is also valid for late time. Through the procedure of renormalization, we extract slow motion from Einstein's equation.

This paper is organized as follows. In Sec. II, we briefly review the gradient expansion. In Sec. III, we introduce the renormalization group method by using two examples. In Sec. IV, the renormalization group method is applied to the solution of the gradient expansion. In Sec. V, we solve the renormalization group equation for several situations. Section VI is devoted to a summary and discussion. We use units in which $c=\hbar=8\pi G=1$ throughout the paper.

II. LONG-WAVELENGTH EXPANSION OF THE EINSTEIN+DUST SYSTEM

We use the synchronous reference frame

$$ds^2=-dt^2+\gamma_{ij}(t,x)dx^idx^j. \qquad (1)$$

Einstein equation's with a dust fluid is

$${}^{(3)}R+K^2-K^k_lK^l_k=2\rho, \qquad (2)$$

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$$K_{,i}-K^j_{i;j}=\sqrt{1+u^k u_k}\rho u_i, \tag{3}$$

$${}^{(3)}R^j_i+\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}(\sqrt{\gamma}K^j_i)=\frac{\rho}{2}(\delta^j_i+2u^j u_i), \tag{4}$$

$$K_{ij}=\frac{1}{2}\frac{\partial\gamma_{ij}}{\partial t}, \tag{5}$$

where K_{ij} is the extrinsic curvature, ρ is the energy density of dust, and u_i is the three-velocity of dust. If we neglect terms with a spatial derivative, the solution can be written

$$\gamma^{(0)}_{ij}=a^2(t)h_{ij}(x), \quad \rho^{(0)}=\rho_0(t), \quad u^{(0)}_i=0, \tag{6}$$

where $h_{ij}(x)$ is an arbitrary function of spatial coordinates (seed metric), and the scale factor $a(t)$ and the energy density $\rho_0(t)$ obey the usual spatially flat FRW equation. Using this as the zeroth order solution, we solve this system by the following gradient expansion [4]:

$$\gamma_{ij}=a^2(t)h_{ij}(x)+[f_2(t)R_{ij}(h)+g_2(t)R(h)h_{ij}]+\cdots, \tag{7}$$

$$u_i=u_3(t)\nabla_i R(h)+\cdots, \tag{8}$$

$$\rho=\rho_0(t)+\rho_2(t)R(h)+\cdots, \tag{9}$$

where $R_{ij}(h)$ is the Ricci tensor of the seed metric h_{ij}. The zeroth order solution is

$$\gamma^{(0)}_{ij}=t^{4/3}h_{ij}(x), \quad u^{(0)}_i=0, \quad \rho^{(0)}=\frac{4}{3t^2}. \tag{10}$$

The solution to the next order is

$$\gamma^{(2)}_{ij}=-\frac{9t^2}{5}\left(R_{ij}(h)-\frac{1}{4}R(h)h_{ij}\right),$$

$$\rho^{(2)}=\frac{3t^{-4/3}}{10}R(h), \quad u^{(3)}_i=0. \tag{11}$$

The solution up to the second order of the spatial gradient can be written as follows:

$$\gamma_{ij}=t^{4/3}\left\{h_{ij}-\frac{9}{5}t^{2/3}\left(R_{ij}(h)-\frac{1}{4}R(h)h_{ij}\right)\right\},$$

$$\rho=\frac{4}{3t^2}\left(1-\frac{9}{20}t^{2/3}R(h)\right), \quad u_i=0. \tag{12}$$

We can see that the perturbation term, which originated from the spatial gradient of the seed metric, grows as the universe expands and finally has the same amplitude as the background term at $t\sim H^{-1}$ when the wavelength of the perturbation equals the Hubble horizon scale. After this time, the wavelength of perturbation becomes smaller than the horizon scale and the long-wavelength expansion breaks down. To make the gradient expansion applicable to the perturbation whose wavelength is smaller than the horizon scale, we use the renormalization group method.

III. RENORMALIZATION GROUP METHOD

The renormalization group method [7–11] improves the long-time behavior of a naive perturbative expansion. We explain the basic concept of the renormalization group method using two examples. The first one is a harmonic oscillator. The equation of motion is

$$\ddot{x}+x=-\epsilon x, \tag{13}$$

where ϵ is a small parameter. We solve this equation perturbatively by expanding the solution with respect to ϵ:

$$x=x_0+\epsilon x_1+\cdots. \tag{14}$$

The solution up to $O(\epsilon)$ becomes

$$x=B_0\cos t+C_0\sin t+\frac{\epsilon}{2}(t-t_0)(C_0\cos t-B_0\sin t)+O(\epsilon^2), \tag{15}$$

where B_0 and C_0 are constants of integration determined by the initial condition at arbitrary time $t=t_0$. This naive perturbation breaks down when $\epsilon(t-t_0)>1$ because of the secular term. To regularize the perturbation series, we introduce an arbitrary time μ, split $t-t_0$ as $t-\mu+\mu-t_0$, and absorb the divergent term containing $\mu-t_0$ into the renormalized counterparts B and C of B_0 and C_0, respectively.

We introduce renormalized constants as follows:

$$B_0=B(\mu)+\epsilon\delta B(\mu,t_0), \quad C_0=C(\mu)+\epsilon\delta C(\mu,t_0), \tag{16}$$

where δB and δC are counterterms that absorb the terms containing $\mu-t_0$ in a naive solution. Substituting Eq. (16) to Eq. (15), we have

$$x=B(\mu)\cos t+C(\mu)\sin t+\epsilon\left\{\delta B\cos t+\delta C\sin t+\frac{1}{2}(t-\mu+\mu-t_0)(C(\mu)\cos t-B(\mu)\sin t)\right\}. \tag{17}$$

δB and δC are chosen as

$$\delta B(\mu,t_0)+\frac{1}{2}(\mu-t_0)C(\mu)=0,$$

$$\delta C(\mu,t_0)-\frac{1}{2}(\mu-t_0)B(\mu)=0. \tag{18}$$

Using the relation $\epsilon\delta B=B_0-B(\mu), \epsilon\delta C=C_0-C(\mu)$,

$$B(t_0)=B(\mu)-\frac{\epsilon}{2}(\mu-t_0)C(\mu),$$

$$C(t_0)=C(\mu)+\frac{\epsilon}{2}(\mu-t_0)B(\mu). \tag{19}$$

These equations define the transformation up to $O(\epsilon)$:

$$\mathcal{R}_{\mu-t_0}: \quad (B(t_0),C(t_0))\rightarrow(B(\mu),C(\mu)).$$

The explicit form of the transformation is

$$\begin{pmatrix} B(\mu) \\ C(\mu) \end{pmatrix}=\mathcal{R}_{\mu-t_0}\begin{pmatrix} B(t_0) \\ C(t_0) \end{pmatrix} = \begin{pmatrix} 1 & \frac{\epsilon}{2}(\mu-t_0) \\ -\frac{\epsilon}{2}(\mu-t_0) & 1 \end{pmatrix}\begin{pmatrix} B(t_0) \\ C(t_0) \end{pmatrix}+O(\epsilon^2). \tag{20}$$

$\mathcal{R}$ satisfies the composition law

$$\begin{aligned}\mathcal{R}_{\mu-\mu_1}\otimes\mathcal{R}_{\mu_1-t_0}&=\begin{pmatrix} 1 & \frac{\epsilon}{2}(\mu-\mu_1) \\ -\frac{\epsilon}{2}(\mu-\mu_1) & 1 \end{pmatrix} \\ &\quad\times\begin{pmatrix} 1 & \frac{\epsilon}{2}(\mu_1-t_0) \\ -\frac{\epsilon}{2}(\mu_1-t_0) & 1 \end{pmatrix} \\ &\quad+O(\epsilon^2) \\ &=\begin{pmatrix} 1 & \frac{\epsilon}{2}(\mu-t_0) \\ -\frac{\epsilon}{2}(\mu-t_0) & 1 \end{pmatrix}+O(\epsilon^2) \\ &=\mathcal{R}_{\mu-t_0},\end{aligned} \tag{21}$$

$\mathcal{R}_0=1$, and $\mathcal{R}_{-\mu}$ is the inverse transformation of $\mathcal{R}_\mu$. So we can conclude that the transformation forms a Lie group up to $O(\epsilon)$. Assuming the properties of the Lie group, we can extend the locally valid expression (19) to a global one, which is valid for arbitrarily large $\mu-t_0$. We apply this transformation to get $(B(\mu),C(\mu))$ at arbitrarily large μ. By differentiating Eq. (19) with respect to μ and setting $t_0=\mu$, we have the renormalization group equations:

$$\frac{\partial B}{\partial\mu}=\frac{\epsilon}{2}C(\mu), \quad \frac{\partial C}{\partial\mu}=-\frac{\epsilon}{2}B(\mu). \tag{22}$$

The renormalized solution becomes

$$x(t)=B(\mu)\cos t+C(\mu)\sin t + \frac{\epsilon}{2}(t-\mu)[C(\mu)\cos t-B(\mu)\sin t]. \tag{23}$$

Solving the renormalization group equation (22) and equating μ and t in Eq. (23) eliminates the secular term and we get the uniformly valid result

$$x(t)=B(0)\cos\left(1+\frac{\epsilon}{2}\right)t+C(0)\sin\left(1+\frac{\epsilon}{2}\right)t. \tag{24}$$

The second example is Einstein's equation for a FRW universe with dust. The spatial component of Einstein's equation is

$$\ddot{\alpha}+\frac{3}{2}(\dot{\alpha})^2=-\frac{\epsilon}{2}e^{-2\alpha}, \tag{25}$$

where $\alpha(t)$ is the logarithm of the scale factor of the universe $a(t)$ and ϵ is the sign of the spatial curvature. The exact solution is given by

$$a(t)=e^{\alpha(t)} =\begin{cases} a_0(1-\cos\eta), & t=a_0(\eta-\sin\eta) \quad \text{for } \epsilon=1, \\ a_0\frac{\eta^2}{2}, & t=a_0\frac{\eta^3}{6} \quad \text{for } \epsilon=0, \\ a_0(\cosh\eta-1), & t=a_0(\sinh\eta-\eta) \quad \text{for } \epsilon=-1. \end{cases} \tag{26}$$

We solve Eq. (25) perturbatively by assuming that the right hand side is small. This is an expansion with respect to the small spatial curvature around the flat universe. By substituting $\alpha=\alpha_0+\epsilon\alpha_1+\cdots$ into Eq. (25), we have the naive solution

$$\alpha=\ln\tau+C_0-\epsilon\frac{9}{20}e^{-2C_0}(\tau-\tau_0)+O(\epsilon^2), \tag{27}$$

where we introduced a new time variable $\tau=t^{2/3}$ and C_0 is a constant of integration determined by the initial condition at $\tau=\tau_0$. The $O(\epsilon)$ term is secular and we regularize this term by introducing arbitrary time μ and renormalized constant $C_0=C(\mu)+\epsilon\delta C(\mu,\tau_0)$:

$$\alpha=\ln\tau+C(\mu)+\epsilon\delta C(\mu,\tau_0) -\epsilon\frac{9}{20}e^{-2C(\mu)}(\tau-\mu+\mu-\tau_0). \tag{28}$$

The counterterm δC is determined so as to absorb the $\mu-\tau_0$ dependent term:

$$\delta C(\mu,\tau_0)-\frac{9}{20}e^{-2C(\mu)}(\mu-\tau_0)=0. \tag{29}$$

This defines the renormalization group transformation $\mathcal{R}_{\mu-\tau_0}: \quad C(\tau_0)\rightarrow C(\mu)$,

$$C(\mu)=C(\tau_0)-\epsilon\frac{9}{20}e^{-2C(\mu)}(\mu-\tau_0), \tag{30}$$

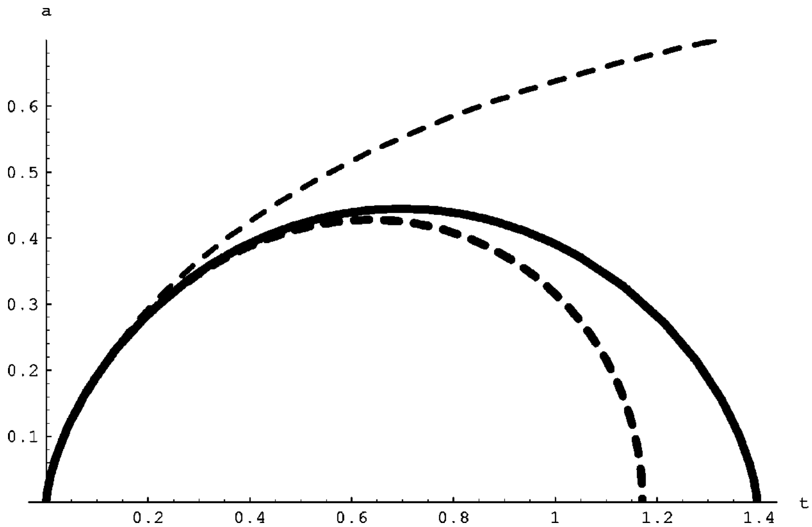

FIG. 1. The evolution of the scale factor for a closed FRW universe with dust. The solid curve is the exact solution, the thin dashed curve is the naive solution, and the thick dashed curve is the renormalized solution.

and this transformation forms the Lie group up to $O(\epsilon)$. So we can have $C(\mu)$ for arbitrarily large values of $\mu-\tau_0$ by assuming the property of the Lie group, and this makes it possible to produce a globally uniform approximated solution of the original equation. The renormalization group equation is

$$\frac{\partial C(\mu)}{\partial \mu}=-\epsilon\frac{9}{20}e^{-2C(\mu)}, \tag{31}$$

and the solution is

$$C(\mu)=\frac{1}{2}\ln\left(c-\frac{9\epsilon}{10}\mu\right), \tag{32}$$

where c is a constant of integration. The renormalized scale factor is given by

$$a(t)=e^{\alpha(t)}=\tau e^{C(\tau)}=t^{2/3}\left(c-\frac{9\epsilon}{10}t^{2/3}\right)^{1/2}. \tag{33}$$

As the zeroth order solution, it is possible to include another integration constant t_0 which defines the origin of cosmic time t. By requiring that the renormalization group transformation form the Lie group, it can be shown that t_0 does not get renormalized. So it is sufficient to consider a solution with the boundary condition $a(t=0)=0$ which fixes the value of t_0 equal to zero. This point is different from the example of a harmonic oscillator, in which case two integration constants B and C both get renormalized.

We compare the renormalized solution (33) with the exact solution (26) and the naive solution (27) for the case of a closed universe ($\epsilon=1$). We choose $a_0=2/9$ and $c=1$. The scale factor of the exact solution has a maximum at $t=2\pi/9$ and goes to zero at $t=4\pi/9$. The naive solution does not show this behavior. The renormalized solution improves the naive solution and reproduces the expanding and contracting feature of the exact solution (Fig. 1).

IV. RENORMALIZATION OF THE LONG-WAVELENGTH SOLUTION

Now we proceed to the long-wavelength solution of Einstein's equation. We renormalize the secular behavior of the three-metric γ_{ij}. By introducing a new time variable $\tau=t^{2/3}$, the long-wavelength solution can be written as

$$\gamma_{ij}=\tau\left\{h_{ij}(x)-\frac{9\tau}{5}\left(R_{ij}(h)-\frac{1}{4}R(h)h_{ij}\right)\right\}. \tag{34}$$

We renormalize the secular term $(9\tau/5)(R_{ij}-\frac{1}{4}Rh_{ij})$. We introduce the initial time τ_0 by re-defining the seed metric h_{ij}, and define the renormalized metric and the counterterm $h_{ij}(x)=h_{ij}(x,\mu)+\delta h_{ij}(x,\mu,\tau_0)$:

$$\begin{aligned}h_{ij}(x)-\frac{9}{5}(\tau-\tau_0)\left(R_{ij}-\frac{1}{4}R(h)h_{ij}\right)\\=h_{ij}(x,\mu)+\delta h_{ij}-\frac{9}{5}(\tau-\mu+\mu-\tau_0)\\\times\left(R_{ij}-\frac{1}{4}R(h)h_{ij}\right).\end{aligned} \tag{35}$$

The counterterm is determined so that it absorbs terms containing $\mu-\tau_0$:

$$\delta h_{ij}(\mu,\tau)-\frac{9}{5}(\mu-\tau_0)\left(R_{ij}(h)-\frac{1}{4}R(h)h_{ij}\right)=0. \tag{36}$$

Using the definition of δh_{ij},

$$\begin{aligned}h_{ij}(x,\mu)=h_{ij}(x,\tau_0)-\frac{9}{5}(\mu-\tau_0)\bigg(R_{ij}(h(\mu))\\-\frac{1}{4}R(h(\mu))h_{ij}(\mu)\bigg).\end{aligned} \tag{37}$$

This equation defines the renormalization group transformation $\mathcal{R}_{\mu-\tau_0}$: $h_{ij}(\tau_0)\to h_{ij}(\mu)$ and this transformation is the Lie group up to $O(\epsilon)$. We therefore can get the value of

$h_{ij}(\mu)$ for arbitrary μ using the relation (37) by assuming the property of the Lie group. The renormalization group equation is obtained by differentiating Eq. (37) with respect to μ and setting $\tau_0=\mu$:

$$\frac{\partial}{\partial\mu}h_{ij}(x,\mu)=-\frac{9}{5}\left[R_{ij}(h(\mu))-\frac{1}{4}R(h(\mu))h_{ij}(\mu)\right], \tag{38}$$

and the renormalized solution is

$$\gamma_{ij}=t^{4/3}h_{ij}(x,t), \tag{39}$$

$$\rho=\frac{4}{3t^2}+\frac{3t^{-4/3}}{10}R(h(x,t)). \tag{40}$$

V. SOLUTION OF THE RENORMALIZATION GROUP EQUATION

We solve the renormalization group equation (38) for some special cases and see how the renormalization group method improves the behavior of the long-wavelength solution.

A. FRW case

The metric is

$$h_{ij}(x,t)=\Omega^2(t)\sigma_{ij}(x),\quad R_{ij}(\sigma)=\frac{1}{3}\sigma_{ij}R(\sigma),$$

where $\sigma_{ij}(x)$ is the metric of three-dimensional maximally symmetric space. In this case, the renormalization group equation reduces to

$$\frac{\partial}{\partial\tau}\Omega^2(\tau)=-\frac{9}{10}k\quad(k=\pm1,0),$$

and the solution is

$$\Omega(t)=\sqrt{c-\frac{9k}{10}t^{2/3}},$$

where c is a constant of integration. The renormalized solution is

$$\gamma_{ij}=t^{4/3}\left(c-\frac{9k}{10}t^{2/3}\right)\sigma_{ij}(x),$$

$$\rho=\frac{4}{3t^2}\left(1+\frac{27}{20}\frac{kt^{2/3}}{c-\frac{9k}{10}t^{2/3}}\right), \tag{41}$$

and the scale factor of the universe is given by

$$a(t)=t^{2/3}\sqrt{c-\frac{9k}{10}t^{2/3}}. \tag{42}$$

This is the same as the solution (33).

B. Spherically symmetric case

In spherical coordinates (r,θ,ϕ), the metric is

$$h_{ij}=\mathrm{diag}(A^2(\tau,r),B^2(\tau,r),B^2(\tau,r)\sin^2\phi). \tag{43}$$

The renormalization group equation (38) becomes

$$2A\frac{\partial A}{\partial\tau}=\frac{9}{5}\left(\frac{A^2}{2B^2}-\frac{A_{,r}B_{,r}}{AB}-\frac{(B_{,r})^2}{2B^2}+\frac{B_{,rr}}{B}\right)\quad(rr\ \text{component}), \tag{44}$$

$$2B\frac{\partial B}{\partial\tau}=\frac{9}{5}\left(-\frac{1}{2}+\frac{(B_{,r})^2}{2A^2}\right)\quad(\theta\theta\ \text{component}). \tag{45}$$

As the special case, Eq. (43) contains a FRW-type metric. So we can expect that the solution of the renormalization equation is given by the generalization of the solution (41). The solution is given by

$$B=\left(1-\frac{9\alpha(r)}{10}\tau\right)^{1/2}\beta(r), \tag{46}$$

$$A=\frac{B_{,r}}{\sqrt{1-\alpha(r)\beta^2(r)}}, \tag{47}$$

where $\alpha(r)$ and $\beta(r)$ are arbitrary functions of r. The renormalized metric and density are

$$ds^2=-dt^2+\frac{(t^{2/3}B_{,r})^2}{1-\alpha\beta^2}dr^2+(t^{2/3}B)^2d\Omega_2^2, \tag{48}$$

$$\rho=\frac{4}{3t^2}+\frac{3}{5t^{4/3}}\frac{3\alpha\beta_{,r}\left(1-\frac{9\alpha}{10}\tau\right)+\alpha_{,r}\beta\left(1-\frac{27\alpha}{20}\tau\right)}{\left(1-\frac{9\alpha}{10}\tau\right)\left[\left(1-\frac{9\alpha}{10}\tau\right)\beta_{,r}-\frac{9}{20}\alpha_{,r}\beta\tau\right]}. \tag{49}$$

By choosing $\beta=r,\alpha=0,\pm1$, we can recover the solution of the FRW case (41). This solution corresponds to the Toleman-Bondi solution [12]

$$ds^2=-dt^2+\frac{R_{,r}(t,r)^2}{1+2E(r)}dr^2+R(t,r)^2d\Omega_2^2, \tag{50}$$

where

$$R(t,r)=\begin{cases} -\dfrac{M(r)}{2E(r)}(1-\cos\eta), \quad t=\dfrac{M(r)}{[-2E(r)]^{3/2}}(\eta-\sin\eta) & \text{for } E(r)<0,\\ \left(\dfrac{9M(r)}{2}\right)^{1/3} t^{2/3} & \text{for } E(r)=0,\\ \dfrac{M(r)}{2E(r)}(\cosh\eta-1), \quad t=\dfrac{M(r)}{[2E(r)]^{3/2}}(\sinh\eta-\eta) & \text{for } E(r)>0, \end{cases} \tag{51}$$

and

$$\rho(t,r)=\frac{2M_{,r}(r)}{R^2 R_{,r}}. \tag{52}$$

$E(r),M(r)$ are arbitrary functions of the radial coordinate r and are connected to the initial distribution of the spatial curvature and the initial distribution of the mass density of dust, respectively. They have the following correspondence with the functions α,β contained in the renormalization group solution (48):

$$2E(r)\leftrightarrow-\alpha(r)\beta^2(r), \quad \frac{9}{2}M(r)\leftrightarrow\beta^3(r). \tag{53}$$

The renormalized solution reproduces well the features of the metric of the spherically symmetric gravitational collapse of dust.

C. Szekeres solution

In Cartesian coordinates (x,y,z), the metric is assumed to be

$$h_{ij}=\mathrm{diag}(1,1,A^2(\tau,x,y,z)). \tag{54}$$

The renormalization group equation (38) reduces to the following three equations:

$$0=-\frac{A_{,xy}}{A} \quad (xy \ \text{component}),$$

$$0=\frac{A_{,yy}-A_{,xx}}{2A} \quad (xx \ \text{component}),$$

$$\frac{\partial A^2}{\partial\tau}=-\frac{1}{2}A(A_{,xx}+A_{,yy}) \quad (zz \ \text{component}). \tag{55}$$

The solution of this equation is

$$A=g(z)(x^2+y^2)+\frac{9}{5}g(z)\tau+c(z), \tag{56}$$

where c,g are arbitrary functions of z. The renormalized metric is

$$ds^2=-dt^2+t^{4/3}\left[dx^2+dy^2+\left(g(z)(x^2+y^2)+\frac{9g(z)}{5}t^{2/3}+c(z)\right)^2dz^2\right]. \tag{57}$$

This is the Szekeres' exact solution [13], which represents one-dimensional gravitational collapse. It is known that the ''naive'' gradient expansion reproduces this solution by including the fourth order spatial gradient [3]. We obtained the solution using the second order spatial gradient with renormalization. In this case, the renormalization procedure much improves the naive solution.

VI. SUMMARY AND DISCUSSION

We have investigated renormalization of the long-wavelength solution of Einstein's equation which has secular behavior. We applied the renormalization group method to improve the long-time behavior of the solution. We introduced the renormalization point $\mu\sim L/H^{-1}$ where L is the physical wavelength of perturbation. For $\mu\ll1(L\gg H^{-1})$, the naive expansion (gradient expansion) approximates well the exact solution. But as the universe expands, $\mu\sim H^{-1}/L \propto t^{1/3}$ becomes large, and at $\mu\sim1(L\sim H^{-1})$, perturbative expansion breaks down. After this time, the wavelength of the perturbation is smaller than the horizon scale. Using a renormalization transformation which has the property of the Lie group, we can renormalize the secular terms caused by the spatial gradient of the background seed metric. The renormalized metric can be extended to a large value of μ, which corresponds to the small scale fluctuation.

We obtained the solution of the renormalization group equation for FRW, spherically symmetric, and Szekeres cases. The behavior of the renormalized solution indicates that they describe the collapsing phase of the system qualitatively well. The renormalization group method is regarded as the procedure of system reduction. This means that the renormalization group equation (38) is a reduced version of the original Einstein's equation and describes the slow motion dynamics of the original equation. We expect the renormalization group equation (38) has physically interesting properties and solutions which are contained in Einstein's equation.

We can look at the renormalization of the long-wavelength solution from the viewpoint of the back reaction problem in cosmology. The naive solution represents the evolution of the perturbation with a fixed background metric. By renormalizing the naive solution, the constants $h_{ij}(x)$ contained in the background solution come to have a time dependence due to the spatial inhomogeneity. So we can investigate how the spatial inhomogeneity affects the ''background'' metric h_{ij} by solving the renormalization group equation. The remarkable feature of the renormalization group approach is that it does not need any spatial averaging which is necessary in the conventional approach to the back reaction problem in cosmology [14].

In this paper, we considered only a dust fluid as the matter field. But the gradient expansion can treat a perfect fluid with pressure and scalar fields. For the scalar field system, although we cannot write down exact solutions of the naive expansion in general, it is possible to define the renormalization group transformation and obtain the renormalized solution formally. It is interesting to apply the renormalization group method to the inflationary model and investigate the effect of the stochastic quantum noise on the background geometry. We can construct the inhomogeneous model of stochastic inflation [15–18] and this is our next subject to be explored.

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